Alan Turing - Systems of Logic Based on Ordinals (1938)

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0.1 Context

0.2 Learned in this study

0.3 Things to explore

1 Overview

2 Notes

Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two faculties, which we may call intuition and ingenuity.

Intuition: making spontaneous judgments which are not the result of conscious trains of reasoning.
Ingenuity: aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings.

Let G be an arithmetic formula that is not provable in the system of arithmetic.

Let G_1 be the incomplete formal system of arithmetic with G as one of its axiom.

Based on the Gödel construction, we can apply this ad infinitum: G_1, G_2, G_3, ...

Let the system of arithmetic that forms the starting point of this infinite progression be called L. The result of adding G to L is called L_1, the result of adding G_1 to L_1 is L_2, and so on. Taken together, the systems in the infinite progression L, L_1, L_2, L_3, ... form a non-constructive logic.

There are a lot of systems in the progression L, L_1, L_2, L_3, ... There is a system that contains the theorems of every one of the systems L_i, where i is a finite ordinal. This system is called L_ω (ω being the first transfinite ordinal number).

L_ω is "bigger" than any of the systems L_i in the sense that any L_i considered, L_ω includes all of its theorems (but not vice versa).

But even L_ω has a true but unprovable G_ω. Adding G_ω to L_ω produces $L_{\omega+1}$. The progression of systems L, L_1, L_2, L_3, ..., L, $L_{\omega+1}$, $L_{\omega+2}$, $L_{\omega+3}$, ... is an example of an ordinal logic.

2.1 o-machines

Turing introduces a new type of machine which is able to determine, given the description of a machine **m** if it is circle-free. In other words, that machine is able to answer to the halting problem.

In the same manner as was demonstrated that no Turing machine can decide if a given Turing machine description number is circle-free, he proves the existence of mathematical problems of the same nature for o-machines. In other words, no o-machine can decide, of arbitrarily selected o-machine description numbers, which are numbers of circle-free o-machine.

2.2 2. Effective calculability. Abbreviation of treatment

- A function is said to be "effectively calculable" if its values can be found by some purely mehanical process.

3 See also

4 References